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EVEN VERTEX EQUITABLE EVEN LABELING FOR PATH RELATED GRAPHS

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Abstract: Let G be a graph with p vertices and q edges and $A = \{0,2,4,\dots,q+1\}$ if q is odd or $A = \{0,2,4,\dots,q\}$ if q is even. A graph G is said to be an even vertex equitable even labeling if there exists a vertex labeling $f: V(G) \to A$ that induces an edge labeling f^* defined by $f^*(uv) = f(u) + f(v)$ for all edges uv such that for all a and b in A, $|v_f(a) - v_f(b)| \leq 1$ and the induced edge labels are $2,4,\dots,2q$, where $v_f(a)$ be the number of vertices v with $f(v) = a$ for $a \in A$. A graph that admits even vertex equitable even labeling is called an even vertex equitable even graph. In this paper, we prove that square of path, $S(P_n \odot K_1)$, $S'(P_n)$,$T(P_n)$, graph obtained by duplication of each vertex by an edge in P_n, quadrilateral snake, $S(Q_n)$, $D(Q_n)$, $A(T_n)$ and $DA(T_n)$ are even vertex equitable even graphs.

Keywords: vertex equitable labeling, even vertex equitable even labeling,

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1. INTRODUCTION

All graphs considered here are simple, finite, connected and undirected. Let $G(V,E)$ be a graph with p vertices and q edges. We follow the basic notations and terminologies of graph theory as in [2]. A graph labeling is an assignment of integers to the vertices or edges or both, subject to certain conditions and a detailed survey of graph labeling can be found in [1]. The concept of vertex equitable labeling was introduced by Lourdusamy and Seenivasan [3]. We introduced the concept of even vertex equitable even labeling in [4].

Definition 1.1: Let G be a graph with p vertices and q edges and $A=\{0,2,4,\dots,q+1\}$ if q is odd or $A=\{0,2,4,\dots,q\}$ if q is even. A graph G is said to be an even vertex equitable even labeling if there exists a vertex labeling $f:V(G)\to A$ that induces an edge labeling f^* defined by $f^*(uv)=f(u)+f(v)$ for all edges uv such that for all a and b in A, $|v_f(a)-v_f(b)|\le 1$ and the induced edge labels are $2,4,\dots,2q$, where $v_f(a)$ be the number of vertices v with $f(v)=a$ for $a\in A$. A graph that admits even vertex equitable even labeling is called an even vertex equitable even graph.

Definition 1.2: For a simple connected graph G the *square of graph* G is denoted by $G^2$and defined as the graph with the same vertex set as of G and two vertices are adjacent in G^2 if they are at a distance 1 or 2 apart in G.

Definition 1.3: The *subdivision of graph* $S(G)$ is obtained from G by subdividing each edge of G with a vertex.

Definition 1.4: The *corona* $G_1 \odot G_2$ of two graphs $G_1(p_1,q_1)$ and $G_2(p_2,q_2)$ is defined as the graph obtained by taking one copy of G_1 and p_1 copies of G_2 and joining the i^{th} vertex of $G_1$with an edge to every vertex in the i^{th} copy of G_2.

Definition 1.5: For a graph G the *splitting graph* $S(G')$ of graph G is obtained by adding a new vertex v'corresponding to each vertex v of G such that $N(v)=N(v')$.

Definition 1.6: For every vertex $v\in V(G)$, the open neighbourhood set $N(v)$ is the set of all vertices adjacent to v in G.

Definition 1.7: Duplication of a vertex v_k by a new edge $e=v_k'v_k''$ in a graph G produces a new graph G' such that $N(v_k')\cap N(v_k'')=v_k$.

Definition 1.8: The *total graph* $T(G)$of a graph G is the graph whose vertex set is $V(G) \cup E(G)$ and two vertices are adjacent whenever they are either adjacent or incident in G.

Definition 1.9: A *quadrilateral snake* Q_n is obtained from a path $u_1, u_2, \dots, u_n$ by joining u_i, $u_i + 1$ to new vertices v_i, w_i respectively and then joining v_i and w_i. That is every edge of the path is replaced by a cycle C_4.

Definition 1.10: A *double quadrilateral snake* $D(Q_n)$ consists of two quadrilateral snakes that have a common path.

Definition 1.11: An *alternate triangular snake* $A(T_n)$ is obtained from a path $u_1, u_2, \dots, u_n$ by joining u_i and $u_i + 1$ (alternatively) to newvertex v_i. That is every alternate edge of a path is replaced by C_3.

Definition 1.12: A *double alternate triangular snake* $DA(T_n)$ consists of two alternate triangular snakes that have a common path. That is, a double alternate triangular snake is obtained from a path $u_1, u_2, \dots, u_n$ by joining u_i and $u_i + 1$ (alternatively) to two new vertices $v_i$and w_i.

2. MAIN RESULTS

Theorem 2.1 The graph P_n^2 is an even vertex equitable even graph.

Proof: Let $u_1, u_2, \dots, u_n$ be the vertices of P_n^2. Then P_n^2 is of order n and size $2n - 3$.

Define $f: V(P_n^2) \to A = \{0,2,4 \dots, 2n - 2\}$ as follows:

$$f(u_i) = 2i - 2 \ ; 1 \leq i \leq n.$$

It can be easily verified that the induced edge labels of P_n^2 are $2,4,6, \dots, 4n - 6$.

Thus,$|v_f(a) - v_f(b)| \leq 1$ for all $a, b \in A$.

Hence the graph P_n^2 is an even vertex equitable even graph.

Theorem 2.2 The graph $S(P_n \odot K_1)$ is an even vertex equitable even graph.

Proof: Let $V(P_n \odot K_1) = \{u_i, v_i : 1 \leq i \leq n\}$ and

$$E(P_n \odot K_1) = \{u_i u_{i+1} : 1 \leq i \leq n - 1\} \cup \{u_i v_i : 1 \leq i \leq n\}.$$

Let v_i' be the newly added vertex between u_i and v_i. Let u_i' be the newly added vertex between u_i and u_{i+1}. Then $S(P_n \odot K_1)$ is of order $4n-1$ and size $4n-2$.

Define $f: V(S(P_n \odot K_1)) \to A = \{0,2,4 \dots ,4n-2\}$ as follows:

$$f(u_i) = \begin{cases} 4i-2 & if\ i\ is\ odd \\ 4i-4 & if\ i\ is\ even \end{cases};\ 1 \le i \le n$$

$$f(v_i) = \begin{cases} 4i-4 & if\ i\ is\ odd \\ 4i-2 & if\ i\ is\ even \end{cases};\ 1 \le i \le n$$

$$f(u_i') = \begin{cases} 4i & if\ i\ is\ odd \\ 4i+2 & if\ i\ is\ even \end{cases};\ 1 \le i \le n-1$$

$$f(v_1') = 2\ ;$$

$$f(v_i') = \begin{cases} 4i-4 & if\ i\ is\ odd \\ 4i-2 & if\ i\ is\ even \end{cases};2 \le i \le n$$

It can be easily verified that the induced edge labels of $S(P_n \odot K_1)$ are $2,4,6, \dots ,8n-4$.

Thus, $|v_f(a) - v_f(b)| \le 1$ for all $a, b \in A$.

Hence the graph $S(P_n \odot K_1)$ is an even vertex equitable even graph.

Theorem 2.3 The splitting graph $S'(P_n)$ is an even vertex equitable even graph.

Proof: Let $u_1, u_2, \dots, u_n$ be the vertices of P_n and $u_1, u_2, \dots, u_n, u_1', u_2', \dots, u_n'$ be the vertices of $S'(P_n)$. Then $S'(P_n)$ is of order $2n$ and size $3(n-1)$.

Define $f: V(S'(P_n)) \to A = \begin{cases} 0,2,4, \dots ,3(n-1)+1 & if\ \ 3(n-1)\ \ is\ odd \\ 0,2,4, \dots ,3(n-1) & if\ \ 3(n-1)\ is\ even \end{cases}$ as follows:

Case (i): n is odd ,$n > 3$.

$$f(u_1) = 0\ ; f(u_2) = 2\ ;$$

$$f(u_{n-1}) = 3(n-1)\ ;$$

$$f(u_n) = 3(n-1) - 2\ ;$$

$$f(u_i) = \begin{cases} 3i-1 & if\ i\ is\ odd \\ 3i-2 & if\ i\ is\ even \end{cases};\ 3 \le i \le n-2$$

$$f(u_1') = 0\ ;\ f(v_2') = 2\ ;$$

$$f(u_i') = \begin{cases} 3i-3 & \text{if } i \text{ is odd} \\ 3i-6 & \text{if } i \text{ is even} \end{cases};\ 3 \le i \le n$$

Case (ii): n is even

$$f(u_i) = \begin{cases} 3i-3 & \text{if } i \text{ is odd} \\ 3i-2 & \text{if } i \text{ is even} \end{cases};\ 1 \le i \le n$$

$$f(u_i') = \begin{cases} 3i-1 & \text{if } i \text{ is odd} \\ 3i-4 & \text{if } i \text{ is even} \end{cases};\ 1 \le i \le n$$

It can be easily verified that the induced edge labels of $S'(P_n)$ are $2,4,6,\dots,6(n-1)$.

Thus, $|v_f(a) - v_f(b)| \le 1$ for all $a, b \in A$.

Hence the graph $S'(P_n)$ is an even vertex equitable even graph.

Theorem 2.4 The total graph $T(P_n)$ is an even vertex equitable even graph.

Proof: Let $u_1, u_2, \dots, u_n$ be the vertices of P_n. Let $V(T(P_n)) = \{u_1, u_2, \dots, u_n, u_1', u_2', \dots, u_n'\}$. Then $T(P_n)$ is of order $2n-1$ and size $4n-5$.

Define $f: V(T(P_n)) \to A = \{0,2,4,\dots,4n-4\}$ as follows:

$$f(u_1) = 0\ ;$$

$$f(u_i) = 4i-6\ ;\ 2 \le i \le n$$

$$f(u_i') = 4i\ ;\ 1 \le i \le n-1$$

It can be easily verified that the induced edge labels of $T(P_n)$ are $2,4,6,\dots,8n-10$.

Thus, $|v_f(a) - v_f(b)| \le 1$ for all $a, b \in A$.

Hence the graph $T(P_n)$ is an even vertex equitable even graph.

Theorem 2.5 The graph obtained by duplication of each vertex by an edge in P_n is an even vertex equitable even graph.

Proof: Let $u_1, u_2, \dots, u_n$ be the vertices of the path P_n and G be the graph obtained by duplication of each vertex u_i of the path P_n by an edge $u_i'u_i''$ for $1 \le i \le n$ at a time.

Let $V(G) = \{u_i, u_i', u_i'' : 1 \leq i \leq n\}$ and
$E(G) = \{u_i u_i', u_i u_i'', u_i' u_i'' : 1 \leq i \leq n\} \cup \{u_i u_{i+1} : 1 \leq i \leq n-1\}$. Then G is of order $3n$ and size $4n-1$. Define $f: V(G) \to A = \{0,2,4, \dots, 4n\}$ as follows:

$$f(u_i) = \begin{cases} 4i-4 & if\ i\ is\ odd \\ 4i & if\ i\ is\ even \end{cases}; 1 \leq i \leq n$$

$$f(u_i') = 4i - 2 \ ; 1 \leq i \leq n$$

$$f(u_i'') = \begin{cases} 4i & if\ i\ is\ odd \\ 4i-4 & if\ i\ is\ even \end{cases}; \ 1 \leq i \leq n$$

It can be easily verified that the induced edge labels of G are $2,4,6, \dots, 8n-2$.

Thus, $|v_f(a) - v_f(b)| \leq 1$ for all $a, b \in A$.

Hence the graph obtained by duplication of each vertex by an edge in P_n is an even vertex equitable even graph.

Theorem 2.6 The quadrilateral snake Q_n is an even vertex equitable even graph.

Proof: The quadrilateral snake is obtained from a path $u_1, u_2, \dots, u_n$. By joining u_i, u_{i+1} to the new vertices v_i, w_i represented and joining v_i and w_i for $1 \leq i \leq n-1$. Then Q_n is of order $3n-2$ and size $4n-4$.

Define $f: V(Q_n) \to A = \{0,2,4, \dots, 4n-4\}$ as follows:

$$f(u_i) = 4i - 4 \ ; 1 \leq i \leq n$$

$$f(v_i) = 4i - 2 \ ; 1 \leq i \leq n-1$$

$$f(w_i) = 4i \ ; 1 \leq i \leq n-1$$

It can be easily verified that the induced edge labels of Q_n are $2,4,6, \dots, 8n-8$.

Thus, $|v_f(a) - v_f(b)| \leq 1$ for all $a, b \in A$.

Hence the graph Q_n is an even vertex equitable even graph.

Theorem 2.7 The subdivision of quadrilateral snake $S(Q_n)$ is an even vertex equitable even graph.

Proof: Let P_n be a path $u_1, u_2, \dots, u_n$.

Let $V(S(Q_n)) = \{v_i, w_i, x_i, y_i, z_i, u_i' : 1 \le i \le n-1\} \cup \{u_i : 1 \le i \le n\}$ and $E(S(Q_n)) = \{u_i u_i', u_i' u_{i+1}, u_i x_i, u_{i+1} y_i, v_i x_i, w_i y_i, v_i z_i, z_i w_i : 1 \le i \le n-1\}$.Then $S(Q_n)$ is of order $7n-6$ and size $8n-8$.

Define $f: V(S(Q_n)) \to A = \{0,2,4, \dots, 8n-8\}$ as follows:

$f(u_1) = 4$;

$f(u_{i+1}) = 8i$; $1 \le i \le n-1$

$f(u_1') = 8$;

$f(u_i') = 8i - 2$; $2 \le i \le n-1$

$f(v_i) = 8i - 6$; $1 \le i \le n-1$

$f(z_1) = 2$;

$f(z_i) = 8i - 2$; $2 \le i \le n-1$

$f(w_1) = 0$;

$f(w_i) = 8i - 4$; $2 \le i \le n-1$

$f(x_1) = 6$;

$f(x_i) = 8i - 6$; $2 \le i \le n-1$

$f(y_1) = 6$;

$f(y_i) = 8i$; $2 \le i \le n-1$

It can be easily verified that the induced edge labels of $S(Q_n)$ are $2,4,6, \dots, 16n-16$.

Thus, $|v_f(a) - v_f(b)| \le 1$ for all $a, b \in A$.

Hence the graph $S(Q_n)$ is an even vertex equitable even graph.

Theorem 2.8 The double quadrilateral snake Q_n is an even vertex equitable even graph.

Proof: The quadrilateral snake is obtained from a path $u_1, u_2, \dots, u_n$.

Let $V(D(Q_n)) = \{u_i : 1 \le i \le n\} \cup \{v_i, w_i, v_i', w_i' : 1 \le i \le n-1\}$ and
$E(D(Q_n)) = \{u_i u_{i+1} : 1 \le i \le n-1\} \cup \{u_{2i-1}v_i, u_{2i-1}v_i', u_{2i}w_i', v_i'w_i', v_iw_i, u_{2i}w_i : 1 \le i \le n-1\}$
. Then Q_n is of order $5n-4$ and size $7n-7$.

Define $f: V(DQ_n) \to A = \begin{cases} 0,2,4,\dots,7n-6 & if\ 7n-7\ is\ odd \\ 0,2,4,\dots,7n-7 & if\ 7n-7\ is\ even \end{cases}$ as follows:

$$f(u_i) = \begin{cases} 7i-7 & if\ i\ is\ odd \\ 7i-6 & if\ i\ is\ even \end{cases}; 1 \le i \le n$$

$$f(v_i) = \begin{cases} 7i-5 & if\ i\ is\ odd \\ 7i-6 & if\ i\ is\ even \end{cases}; 1 \le i \le n-1$$

$$f(w_i) = \begin{cases} 7i-3 & if\ i\ is\ odd \\ 7i-4 & if\ i\ is\ even \end{cases}; 1 \le i \le n-1$$

$$f(v_i') = \begin{cases} 7i-3 & if\ i\ is\ odd \\ 7i-2 & if\ i\ is\ even \end{cases}; 1 \le i \le n-1$$

$$f(w_i') = \begin{cases} 7i-1 & if\ i\ is\ odd \\ 7i & if\ i\ is\ even \end{cases}; 1 \le i \le n-1$$

It can be easily verified that the induced edge labels of DQ_n are $2,4,6,\dots,14n-14$.

Thus, $|v_f(a) - v_f(b)| \le 1$ for all $a, b \in A$.

Hence the graph DQ_n is an even vertex equitable even graph.

Theorem 2.9 An alternate triangular snake $A(T_n)$ is an even vertex equitable even graph.

Proof: Let $u_1, u_2, \dots, u_n$ be the vertices of path P_n. The graph $A(T_n)$ is obtained by joining the vertices u_i, u_{i+1} (alternately) to new vertex v_i, $1 \le i \le n-1$ for even n and $1 \le i \le n-2$ for odd n.

Let $V(A(T_n)) = \{u_i : 1 \le i \le n\} \cup \left\{v_i : 1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor\right\}$ and
$E(A(T_n)) = \{u_i u_{i+1} : 1 \le i \le n-1\} \cup \left\{u_{2i-1}v_i : 1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor\right\} \cup \left\{u_{2i}v_i : 1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor\right\}$. Then

$$|V(A(T_n))| = \begin{cases} \frac{3n-1}{2} & if\ \ n\ is\ odd \\ \frac{3n}{2} & if\ \ n\ is\ even \end{cases}$$

$$|E(A(T_n))| = \begin{cases} 2n-2 & if\ \ n\ is\ odd \\ 2n-1 & if\ \ n\ is\ even \end{cases}$$

Define $f: V(A(T_n)) \to A = \begin{cases} 0,2,4,\dots,2n & if\ \ 2n-1\ is\ odd \\ 0,2,4,\dots,2n-2 & if\ \ 2n-2\ is\ even \end{cases}$ as follows:

$$f(u_{2i-1}) = 4i - 4\ ;\ 1 \le i \le \left\lceil \frac{n}{2} \right\rceil$$

$$f(u_{2i}) = 4i\ ;\ 1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor$$

$$f(v_i) = 4i - 2\ ; 1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor$$

It can be easily verified that the induced edge labels of $A(T_n)$ are $2,4,6,\dots,4n-2$ if n is even and $2,4,6,\dots,4n-4$ if n is odd.

Thus, $|v_f(a) - v_f(b)| \le 1$ for all $a, b \in A$.

Hence the graph $A(T_n)$ is an even vertex equitable even graph.

Theorem 2.10 The double alternate triangular snake $DA(T_n)$ is an even vertex equitable even graph.

Proof: Let $u_1, u_2, \dots, u_n$ be the vertices of path P_n.

Let $V(DA(T_n)) = \{u_i : 1 \le i \le n\} \cup \left\{v_i, w_i : 1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor\right\}$ and $E(DA(T_n)) = \{u_i u_{i+1} : 1 \le i \le n-1\} \cup \left\{u_{2i-1} v_i, u_{2i-1} w_i : 1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor\right\} \cup \left\{u_{2i} v_i, u_{2i} w_i : 1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor\right\}$

. Then

$$|V(DA(T_n))| = \begin{cases} 2n-1 & if\ \ n\ is\ odd \\ 2n & if\ \ n\ is\ even \end{cases}$$

$$|E(DA(T_n))| = \begin{cases} 3n-3 & if\ \ n\ is\ odd \\ 3n-1 & if\ \ n\ is\ even \end{cases}$$

Define $f: V(DA(T_n)) \to A = \begin{cases} 0,2,4,\dots,3n & if\ \ 3n-1\ is\ odd \\ 0,2,4,\dots,3n-3 & if\ 3n-3\ \ is\ even \end{cases}$ as follows:

$$f(u_i) = \begin{cases} 3i-3 & if\ \ i\ is\ odd \\ 3i & if\ \ i\ \ is\ even \end{cases};\ 1 \le i \le n$$

$$f(v_i) = 6i - 4\ ;1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor$$

$$f(w_i) = 6i - 2\ ;1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor$$

It can be easily verified that the induced edge labels of $DA(T_n)$ are $2,4,6, \dots ,6n-2$ if n is even and $2,4,6, \dots ,6n-6$ if n is odd.

Thus, $\left|v_f(a) - v_f(b)\right| \le 1$ for all $a, b \in A$.

Hence the graph $DA(T_n)$ is an even vertex equitable even graph.

REFERENCES

1. J. A. Gallian, "A Dyamic Survey of Graph Labeling",*The Electronic J.Combin.,*17 (2015) #DS6.
2. F. Harary, Graph Theory, Addison-wesley, Reading, Mass 1972.
3. A. Lourdusamy and M. Seenivasan, "Vertex Equitable labeling of graphs", *Journal of Discrete Mathematical Sciences & Cryptography,* 11(6) (2008) 727-735.
4. A. Lourdusamy, J. Shobana Mary and F. Patrick, "Even Vertex Equitable Even Labeling", (*Submitted for publication*)
5. S. K. Vaidya and N. B. Vyas, "Even Mean Labeling for Path and Bistar related Graphs", *International Journal of Graph Theory,* 1(4) (2013) 122-130.
6. S. K. Vaidya and N. B. Vyas, "Product Cordial Labeling for Alternate Snake Graphs", *Malaya Journal of Matematik,* 2(3) (2014) 188-196.